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1 Introduction

Exponential sums of various forms have been considered by many authors. In particular, Kim [1] evaluated the exponential sum of the form

$$\sum_{g\in \mathrm{Sp}(2n,q)} \lambda(\operatorname{tr} g), \tag{1.1}$$

where λ is a nontrivial additive character of $\mathbb{F}_q$. Later, in [2] he generalized the result for the sum

$$\sum_{g\in \mathrm{Sp}(2n,q)} \lambda((\operatorname{tr} g)^r), \tag{1.2}$$

where r is a positive integer. The aim of this thesis is to point out that in fact we can apply the methods used for evaluation of previous sums for the generic (i.e., not adorned by characters) sum

$$\sum_{g\in \mathrm{Sp}(2n,q)} f(\operatorname{tr} g), \tag{1.3}$$

where f is an arbitrary complex valued function defined on $\mathbb{F}_q$. Thus it turns out that we can obtain the formulas for (1.1) and (1.2) as special cases of (1.3).

We also present an explicit formula obtained on the same principle for the generic sum

$$\sum_{g\in \mathrm{GSp}(2n,q)} e(\det g) f(\operatorname{tr} g), \tag{1.4}$$

where e, f are arbitrary complex valued functions defined on $\mathbb{F}_q$. Specializing the functions e and f, we can derive formulas for the exponential sums

$$\sum_{g\in \mathrm{GSp}(2n,q)} \chi(\det g)\lambda(\operatorname{tr} g) \quad \text{and} \quad \sum_{g\in \mathrm{GSp}(2n,q)} \lambda((\operatorname{tr} g)^r), \tag{1.5}$$

where χ is a nontrivial multiplicative character of $\mathbb{F}_q$, λ a nontrivial additive character of $\mathbb{F}_q$, and r a positive integer. Remember that the formulas for (1.5) were derived in [1] and [2] respectively.

These generic sums have some interesting applications. As an example, we will use the formula of (1.4) to count the number of elements in $\mathrm{GSp}(2n, q)$ of given trace and determinant.

In section 2, we prepare ourselves for evaluation. In section 3, we take steps to derive the explicit formula of (1.3). In section 4, we examine consequences and applications of our result.

2 Preliminaries

Here we collect definitions, facts, and notations for easy reference. The notations established here will be used later without further comment. We use the same notations as in [1] and [2], with some slight modifications for convenience.

Let $J = \left[\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right]$. $\mathbb{F}_q$ denotes the finite field with q elements. Recall that the symplectic group $\mathrm{Sp}(2n,q)$ over $\mathbb{F}_q$ and the symplectic similitude group $\mathrm{GSp}(2n,q)$ over $\mathbb{F}_q$ are respectively defined by

$$\begin{aligned} \mathrm{Sp}(2n,q) &= \left\{ g \in \mathrm{GL}(2n,q) \mid {}^t gJg = J \right\}, \\ \mathrm{GSp}(2n,q) &= \left\{ g \in \mathrm{GL}(2n,q) \mid {}^t gJg = \alpha J \text{ for some } \alpha \in \mathbb{F}_q^\times \right\}. \end{aligned}$$

We have a maximal parabolic subgroup of $\mathrm{Sp}(2n,q)$

$$P = P(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \;\middle|\; \begin{matrix} A \in \mathrm{GL}(n,q), \\ {}^t B = B \end{matrix} \right\}.$$

We let for $0 \le b \le n$,

$$A_b = A_b(2n,q) = \left\{ g \in P(2n,q) \mid \sigma_b g \sigma_b^{-1} \in P(2n,q) \right\},$$

where

$$\sigma_b = \begin{bmatrix} 0 & 0 & 1_b & 0 \\ 0 & 1_{n-b} & 0 & 0 \\ -1_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-b} \end{bmatrix}.$$

Now the Bruhat decomposition of $\mathrm{Sp}(2n,q)$ says

$$\mathrm{Sp}(2n,q) = \coprod_{b=0}^{n} P\sigma_b P = \coprod_{b=0}^{n} P\sigma_b (A_b \backslash P).$$

For the sake of clarity of later exposition, we do a computation in advance. Let $g \in P(2n,q)$. Then an easy computation shows

$$\mathrm{tr}(g\sigma_b) = -\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^t B_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22},$$

where we let

$$g = \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad {}^tA^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

Here each of $A, B, {}^tA^{-1}$ is divided into four blocks of suitable size for block multiplication of g and σ_b. Observe that when $A_{12} = 0$, we have $E_{22} = A_{22}^{-1}$.

Lastly we list some counts. Let g_n be the number of $n \times n$ nonsingular matrices over $\mathbb{F}_q$, and a_n the number of $n \times n$ nonsingular alternating matrices over $\mathbb{F}_q$. We define $g_0 = a_0 = 1$. Then

$$\begin{aligned} g_n &= \prod_{j=0}^{n-1} (q^n - q^j) = q^{\frac{n^2-n}{2}} \prod_{j=1}^{n} (q^j - 1), \\ a_n &= \begin{cases} q^{\frac{n}{2}(\frac{n}{2}-1)} \prod_{j=1}^{\frac{n}{2}} (q^{2j-1} - 1) & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}, \end{cases} \\ |A_b \backslash P| &= q^{\frac{b^2+b}{2}} \begin{bmatrix} n \\ b \end{bmatrix}_q . \end{aligned}$$

Here the q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} \frac{q^{n-j} - 1}{q^{r-j} - 1}.$$

See [1] and [2] for these.

3 Evaluation

Our major work is simply to check that the steps taken by Kim to derive his formulas for (1.1) and (1.2) are equally applicable to the generic sum (1.3). We begin with the Bruhat decomposition of $\mathrm{Sp}(2n,q)$. We have

$$\begin{aligned}\sum_{g\in \mathrm{Sp}(2n,q)} f(\operatorname{tr} g) &= \sum_{b=0}^{n} \sum_{g\in P\sigma_b(A_b\backslash P)} f(\operatorname{tr} g) \\ &= \sum_{b=0}^{n} |A_b\backslash P| \sum_{g\in P} f(\operatorname{tr}(g\sigma_b)).\end{aligned} \tag{3.1}$$

For the second equality, observe that for $h \in P$,

$$\sum_{g\in P} f(\operatorname{tr}(g\sigma_b h)) = \sum_{g\in P} f(\operatorname{tr}(hg\sigma_b)) = \sum_{g\in P} f(\operatorname{tr}(g\sigma_b)).$$

We proceed to evaluate

$$\sum_{g\in P} f(\operatorname{tr}(g\sigma_b))$$

for $0 \le b \le n$. For the extreme cases $b = 0$ and $b = n$, we get respectively

$$\sum_{g\in P} f(\operatorname{tr}(g\sigma_0)) = \sum_{g\in P} f(\operatorname{tr} g) = q^{\frac{n^2+n}{2}} \sum_{g\in \mathrm{GL}(n,q)} f(\operatorname{tr} g + \operatorname{tr} g^{-1}) \tag{3.2}$$

and

$$\begin{aligned}\sum_{g\in P} f(\operatorname{tr}(g\sigma_n)) = \sum_{g\in P} f(\operatorname{tr} gJ) &= \sum_{\substack{A\,\text{alt.}\\ B\,\text{sym.}}} f(-\operatorname{tr} AB) + \sum_{\substack{A\,\text{not alt.}\\ B\,\text{sym.}}} f(-\operatorname{tr} AB) \\ &= q^{\frac{n^2+n}{2}} a_n f(0) + q^{\frac{n^2+n}{2}-1}(g_n - a_n) \sum_{\gamma\in\mathbb{F}_q} f(\gamma).\end{aligned} \tag{3.3}$$

Observe that in (3.3), $\operatorname{tr} AB$ is constantly 0 if A is alternating, but runs multiple times over $\mathbb{F}_q$ if not.

For the case $0 < b < n$, we get

$$\begin{aligned}
\sum_{g\in P} f(\mathrm{tr}(g\sigma_b)) &= \sum_{\substack{A\in \mathrm{GL}(n,q)\\ B\,\mathrm{sym.}}} f(-\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^t B_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22}) \\
&= \sum_{A_{12}\neq 0} \cdots + \sum_{\substack{A_{12}=0\\ A_{11}\ \mathrm{alt.}}} \cdots + \sum_{\substack{A_{12}=0\\ A_{11}\ \mathrm{not\ alt.}}} \cdots \\
&= q^{\frac{n^2+n}{2}-1}\big(g_n - a_b g_{n-b} q^{b(n-b)}\big) \sum_{\gamma\in\mathbb{F}_q} f(\gamma) \\
&\quad + a_b q^{\frac{n^2+n}{2}+b(n-b)} \sum_{g\in \mathrm{GL}(n-b,q)} f(\operatorname{tr} g + \operatorname{tr} g^{-1}).
\end{aligned} \tag{3.4}$$

Combining (3.2)–(3.4) with (3.1), we reach to this intermediate result

$$\begin{aligned}
\sum_{g\in \mathrm{Sp}(2n,q)} f(\operatorname{tr} g) = \sum_{b=0}^{n} |A_b\backslash P| \Big\{ & q^{\frac{n^2+n}{2}-1}\big(g_n - a_b g_{n-b} q^{b(n-b)}\big) \sum_{\gamma\in\mathbb{F}_q} f(\gamma) \\
&+ a_b q^{\frac{n^2+n}{2}+b(n-b)} \sum_{g\in \mathrm{GL}(n-b,q)} f(\operatorname{tr} g + \operatorname{tr} g^{-1}) \Big\}.
\end{aligned} \tag{3.5}$$

Here we adopted the convention that $\sum_{g\in \mathrm{GL}(0,q)} f(\operatorname{tr} g + \operatorname{tr} g^{-1}) = f(0)$.

Hence our problem is reduced to the evaluation of the sum

$$\sum_{g\in \mathrm{GL}(n,q)} f(\operatorname{tr} g + \operatorname{tr} g^{-1}). \tag{3.6}$$

The following evaluation of (3.6) is due to Kim. Recall that the ordinary Kloosterman sum $K(\lambda;\sigma,\tau)$ is defined by $K(\lambda;\sigma,\tau) = \sum_{\alpha\in\mathbb{F}_q^\times} \lambda(\sigma\alpha + \tau\alpha^{-1})$. First we state a simplified version of Theorem 4.3 in [1].

Theorem 3.1. *Let us define* $K_{\mathrm{GL}(n,q)}(\lambda;\sigma,\tau) = \sum_{g\in \mathrm{GL}(n,q)} \lambda(\sigma \operatorname{tr} g + \tau \operatorname{tr} g^{-1})$ *for a nontrivial additive character* λ *of* $\mathbb{F}_q$ *and* $\sigma,\tau \in \mathbb{F}_q^\times$. *Then we have*

$$K_{\mathrm{GL}(n,q)}(\lambda;\sigma,\tau) = q^{\frac{n^2-n}{2}} \sum_{l=0}^{[n/2]} q^l K(\lambda;\sigma,\tau)^{n-2l} \sum \prod_{\nu=1}^{l} (q^{n+1-(\nu+j_\nu)} - 1),$$

where the unspecified sum is taken over all tuples $(j_1,\ldots,j_l)$ *with* $0 < j_1 < \cdots < j_l < n-l+1$ *(we take it as 1 if* $l = 0$*).*

Now pick a nontrivial additive character λ of $\mathbb{F}_q$. Then we have

$$
\begin{aligned}
&\sum_{g\in \mathrm{GL}(n,q)} f(\operatorname{tr} g+\operatorname{tr} g^{-1})\\
&\qquad = \sum_{\gamma\in\mathbb{F}_q} |\{\, g\in \mathrm{GL}(n,q) \mid \operatorname{tr} g+\operatorname{tr} g^{-1}=\gamma \,\}|\, f(\gamma)\\
&\qquad = \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\sum_{\delta\in\mathbb{F}_q}\sum_{g\in \mathrm{GL}(n,q)} \lambda(\delta(\operatorname{tr} g+\operatorname{tr} g^{-1}-\gamma))f(\gamma)\\
&\qquad = \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\sum_{\delta\in\mathbb{F}_q^\times} K_{\mathrm{GL}(n,q)}(\lambda;\delta,\delta)\lambda(-\gamma\delta)f(\gamma)+\frac{1}{q}g_n\sum_{\gamma\in\mathbb{F}_q} f(\gamma) \qquad (3.7)\\
&\qquad = q^{\frac{n^2-n}{2}}\sum_{l=0}^{[n/2]} q^l \sum \prod_{\nu=1}^{l}(q^{n+1-(\nu+j_\nu)}-1)\\
&\qquad\qquad \times \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\sum_{\delta\in\mathbb{F}_q^\times} K(\lambda;\delta,\delta)^{n-2l}\lambda(-\gamma\delta)f(\gamma)\\
&\qquad\quad + q^{\frac{n^2-n}{2}-1}\prod_{j=1}^{n}(q^j-1)\sum_{\gamma\in\mathbb{F}_q} f(\gamma),
\end{aligned}
$$

where the unspecified sum is taken over all tuples $(j_1,\ldots,j_l)$ with $0<j_1<\cdots<j_l<n-l+1$.

For clarity, we evaluate the subexpression of (3.7) apart.

$$
\begin{aligned}
&\frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\sum_{\delta\in\mathbb{F}_q^\times} K(\lambda;\delta,\delta)^{n-2l}\lambda(-\gamma\delta)f(\gamma)\\
&\quad = \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\sum_{\delta\in\mathbb{F}_q^\times}\Big(\sum_{\alpha\in\mathbb{F}_q^\times}\lambda(\delta\alpha+\delta\alpha^{-1})\Big)^{n-2l}\lambda(-\gamma\delta)f(\gamma)\\
&\quad = \frac{1}{q}\sum_{\gamma\in\mathbb{F}_q}\sum_{\delta\in\mathbb{F}_q}\sum \lambda(\delta(\alpha_1+\alpha_1^{-1}+\cdots+\alpha_{n-2l}+\alpha_{n-2l}^{-1}-\gamma))f(\gamma) \qquad (3.8)\\
&\qquad -\frac{1}{q}(q-1)^{n-2l}\sum_{\gamma\in\mathbb{F}_q} f(\gamma)\\
&\quad = \sum f(\alpha_1+\alpha_1^{-1}+\cdots+\alpha_{n-2l}+\alpha_{n-2l}^{-1})-\frac{1}{q}(q-1)^{n-2l}\sum_{\gamma\in\mathbb{F}_q} f(\gamma),
\end{aligned}
$$

where the unspecified sums are taken over $\alpha_1,\ldots,\alpha_{n-2l}\in\mathbb{F}_q^\times$.

Let $M_m(f;\sigma,\tau)$ denote

$$M_m(f;\sigma,\tau) = \sum_{\alpha_1,\dots,\alpha_m \in \mathbb{F}_q^\times} f(\sigma\alpha_1 + \tau\alpha_1^{-1} + \cdots + \sigma\alpha_m + \tau\alpha_m^{-1})$$

with the convention that $M_0(f;\sigma,\tau) = f(0)$. Putting (3.8) into (3.7), we obtain the next theorem.

Theorem 3.2. *Let f be a complex valued function defined on $\mathbb{F}_q$. Then*

$$\begin{aligned}
&\sum_{g\in \mathrm{GL}(n,q)} f(\operatorname{tr} g + \operatorname{tr} g^{-1}) \\
&\qquad = q^{\frac{n^2-n}{2}} \sum_{l=0}^{[n/2]} q^l M_{n-2l}(f;1,1) \sum \prod_{\nu=1}^{l} (q^{n+1-(\nu+j_\nu)} - 1) \\
&\qquad\quad - q^{\frac{n^2-n}{2}-1} \sum_{l=0}^{[n/2]} q^l (q-1)^{n-2l} \sum \prod_{\nu=1}^{l} (q^{n+1-(\nu+j_\nu)} - 1) \sum_{\gamma\in\mathbb{F}_q} f(\gamma) \\
&\qquad\quad + q^{\frac{n^2-n}{2}-1} \prod_{j=1}^{n} (q^j - 1) \sum_{\gamma\in\mathbb{F}_q} f(\gamma),
\end{aligned} \tag{3.9}$$

where the unspecified sums are taken over all tuples $(j_1,\dots,j_l)$ with $0 < j_1 < \cdots < j_l < n-l+1$ (we take them as 1 if $l=0$).

Our main theorem, an explicit formula of (1.3), now follows from (3.5) and the above theorem, which we postpone to state until the next concluding section.

4 Conclusion and application

Here is our main theorem.

Theorem 4.1. *Let f be a complex valued function defined on $\mathbb{F}_q$. Then we have*

$$
\begin{aligned}
&\sum_{g\in \mathrm{Sp}(2n,q)} f(\mathrm{tr}\, g) \\
&\quad = q^{n^2}\sum_{b=0}^{[n/2]} q^{b^2+b}\begin{bmatrix} n \\ 2b\end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1)\sum_{l=0}^{[(n/2)-b]} q^l M_{n-2b-2l}(f;1,1) \\
&\qquad\quad \times \sum \prod_{\nu=1}^{l}(q^{n-2b+1-(\nu+j_\nu)}-1) \qquad\qquad (4.1)\\
&\qquad + q^{n^2-1}\Big\{-\sum_{b=0}^{[n/2]} q^{b^2+b}\begin{bmatrix} n \\ 2b\end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1)\sum_{l=0}^{[(n/2)-b]} q^l (q-1)^{n-2b-2l} \\
&\qquad\quad \times \sum \prod_{\nu=1}^{l}(q^{n-2b+1-(\nu+j_\nu)}-1) + \prod_{j=1}^{n}(q^{2j}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q} f(\gamma),
\end{aligned}
$$

where the unspecified sums are taken over all tuples $(j_1,\ldots,j_l)$ with $0<j_1<\cdots<j_l<n-2b-l+1$ (we take them as 1 when $l=0$).

Let us examine what we achieved in the theorem 4.1. Let λ be a nontrivial additive character of $\mathbb{F}_q$. Note that $M_m(\lambda;1,1)$ is the m-th power of the ordinary Kloosterman sum $K(\lambda;1,1)$. Hence, by substituting f with λ in (4.1), we obtain

$$
\begin{aligned}
\sum_{g\in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g) &= q^{n^2}\sum_{b=0}^{[n/2]} q^{b^2+b}\begin{bmatrix} n \\ 2b\end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1)\sum_{l=0}^{[(n/2)-b]} q^l K(\lambda;1,1)^{n-2b-2l} \\
&\qquad \times \sum \prod_{\nu=1}^{l}(q^{n-2b+1-(\nu+j_\nu)}-1),
\end{aligned}
$$

where the unspecified sum is understood as in the theorem 4.1. Observe that the second term of (4.1) disappeared since we have

$$\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma)=0$$

for a nontrivial additive character λ of $\mathbb{F}_q$. This is Theorem A in [1] with a slight simplification.

Let r be a positive integer and λ a nontrivial additive character of $\mathbb{F}_q$. We substitute f with the function defined by $\alpha\mapsto\lambda(\alpha^r)$ for $\alpha\in\mathbb{F}_q$. Then we obtain from (4.1)

$$\begin{aligned}
&\sum_{g\in\mathrm{Sp}(2n,q)}\lambda((\operatorname{tr} g)^r)\\
&\qquad= q^{n^2}\sum_{b=0}^{[n/2]} q^{b^2+b}\begin{bmatrix} n\\ 2b\end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\sum_{l=0}^{[(n/2)-b]} q^l MK_{n-2b-2l}(\lambda^r;1,1)\\
&\qquad\quad\times\sum\prod_{\nu=1}^{l}(q^{n-2b+1-(\nu+j_\nu)}-1)\\
&\qquad\quad+q^{n^2-1}\Big\{-\sum_{b=0}^{[n/2]} q^{b^2+b}\begin{bmatrix} n\\ 2b\end{bmatrix}_q\prod_{j=1}^{b}(q^{2j-1}-1)\sum_{l=0}^{[(n/2)-b]} q^l(q-1)^{n-2b-2l}\\
&\qquad\quad\times\sum\prod_{\nu=1}^{l}(q^{n-2b+1-(\nu+j_\nu)}-1)+\prod_{j=1}^{n}(q^{2j}-1)\Big\}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r),
\end{aligned}$$

where the unspecified sums are understood as in the theorem 4.1. Here $MK_m(\lambda^r;\sigma,\tau)$ denotes

$$MK_m(\lambda^r;\sigma,\tau)=\sum_{\gamma_1,\dots,\gamma_m\in\mathbb{F}_q^\times}\lambda((\sigma\gamma_1+\tau\gamma_1^{-1}+\cdots+\sigma\gamma_m+\tau\gamma_m^{-1})^r)$$

as defined in [2]. This is Theorem A in [2], slightly simplified.

We now turn our attention to the generic sum (1.4). Essentially the same procedure produces the next theorem.

Theorem 4.2. *Let e, f be complex valued functions defined on $\mathbb{F}_q$. Then we have*

$$\begin{aligned}
&\sum_{g\in \mathrm{GSp}(2n,q)} e(\det g) f(\operatorname{tr} g) \\
&\quad = q^{n^2} \sum_{b=0}^{[n/2]} q^{b^2+b} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \sum_{l=0}^{[(n/2)-b]} q^l \sum_{\alpha\in\mathbb{F}_q^\times} e(\alpha^n) M_{n-2b-2l}(f;1,\alpha) \\
&\qquad\quad \times \sum \prod_{\nu=1}^{l} (q^{n-2b+1-(\nu+j_\nu)}-1) \\
&\qquad + q^{n^2-1}\Big\{ -\sum_{b=0}^{[n/2]} q^{b^2+b} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \sum_{l=0}^{[(n/2)-b]} q^l (q-1)^{n-2b-2l} \\
&\qquad\quad \times \sum \prod_{\nu=1}^{l} (q^{n-2b+1-(\nu+j_\nu)}-1) + \prod_{j=1}^{n}(q^{2j}-1)\Big\} \sum_{\alpha\in\mathbb{F}_q^\times} e(\alpha^n) \sum_{\gamma\in\mathbb{F}_q} f(\gamma),
\end{aligned}$$

where the unspecified sums are taken over all tuples $(j_1, \dots, j_l)$ with $0 < j_1 < \cdots < j_l < n-2b-l+1$ (we take them as 1 when $l=0$).

With e, f substituted by a nontrivial multiplicative character χ of $\mathbb{F}_q$ and a nontrivial additive character λ of $\mathbb{F}_q$ respectively, we obtain Theorem B of [1]

$$\begin{aligned}
&\sum_{g\in \mathrm{GSp}(2n,q)} \chi(\det g) \lambda(\operatorname{tr} g) \\
&\quad = q^{n^2} \sum_{b=0}^{[n/2]} q^{b^2+b} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \sum_{l=0}^{[(n/2)-b]} q^l \sum_{\alpha\in\mathbb{F}_q^\times} \chi^n(\alpha) K(\lambda;1,\alpha)^{n-2b-2l} \\
&\qquad \times \sum \prod_{\nu=1}^{l} (q^{n-2b+1-(\nu+j_\nu)}-1),
\end{aligned}$$

where the unspecified sum is understood as in the theorem 4.2.

Let r be a positive integer and λ a nontrivial additive character of $\mathbb{F}_q$. With e identically 1 and f substituted by the function defined by $\alpha \mapsto \lambda(\alpha^r)$

for $\alpha \in \mathbb{F}_q$, we obtain Theorem B in [2]

$$\sum_{g \in \mathrm{GSp}(2n,q)} \lambda((\operatorname{tr} g)^r)$$

$$= q^{n^2} \sum_{b=0}^{[n/2]} q^{b^2+b} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1) \sum_{l=0}^{[(n/2)-b]} q^l \sum_{\alpha \in \mathbb{F}_q^\times} MK_{n-2b-2l}(\lambda^r; 1, \alpha)$$

$$\times \sum \prod_{\nu=1}^{l} (q^{n-2b+1-(\nu+j_\nu)}-1)$$

$$+ q^{n^2-1} \Big\{ - \sum_{b=0}^{[n/2]} q^{b^2+b} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1) \sum_{l=0}^{[(n/2)-b]} q^l (q-1)^{n-2b-2l}$$

$$\times \sum \prod_{\nu=1}^{l} (q^{n-2b+1-(\nu+j_\nu)}-1) + \prod_{j=1}^{n} (q^{2j}-1) \Big\} (q-1) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r),$$

where the unspecified sums are understood as in the theorem 4.2.

Now we will use the theorem 4.2 to count the number of elements of $\mathrm{GSp}(2n,q)$ of given trace and determinant. We define for $\zeta \in \mathbb{F}_q^\times, \eta \in \mathbb{F}_q$,

$$C(\zeta, \eta) = |\{ g \in \mathrm{GSp}(2n,q) \mid \det g = \zeta, \operatorname{tr} g = \eta \}|.$$

Then we clearly have

$$C(\zeta, \eta) = \sum_{g \in \mathrm{GSp}(2n,q)} e(\det g) f(\operatorname{tr} g) \tag{4.2}$$

if we define the functions e and f by

$$\begin{cases} e(\alpha) = 1 & \text{if } \alpha = \zeta, \\ e(\alpha) = 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \begin{cases} f(\alpha) = 1 & \text{if } \alpha = \eta, \\ f(\alpha) = 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

for $\alpha \in \mathbb{F}_q^\times$. Thus we have the following corollary.

Corollary. *Let $\zeta \in \mathbb{F}_q^\times, \eta \in \mathbb{F}_q$. Then we have*

$$C(\zeta, \eta) = q^{n^2-1} \sum_{b=0}^{[n/2]} q^{b^2+b} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1) \sum_{l=0}^{[(n/2)-b]} q^l A_{n-2b-2l}$$

$$\times \sum \prod_{\nu=1}^{l} (q^{n-2b+1-(\nu+j_\nu)}-1) + q^{n^2-1} \prod_{j=1}^{n} (q^{2j}-1) R,$$

where the unspecified sum is taken over all tuples $(j_1, \ldots, j_l)$ *with* $0 < j_1 < \cdots < j_l < n-2b-l+1$ *(we take them as 1 when* $l = 0$*). Here* R *is the number of* n*-th roots of* ζ *in* $\mathbb{F}_q$*, and* A_m *is given by*

$$A_m = q \sum_{\alpha \in \mathbb{F}_q^\times} \sum_{\alpha_1, \ldots, \alpha_m \in \mathbb{F}_q^\times} h(\alpha^n, \alpha_1 + \alpha\alpha_1^{-1} + \cdots + \alpha_m + \alpha\alpha_m^{-1}) - (q-1)^m R,$$

where $h(x,y) = 1$ *if* $(x,y) = (\zeta, \eta)$*, 0 otherwise; and the inner sum is understood as* $h(\alpha^n, 0)$ *for* $m = 0$*.*

We show in the next page the tables of $C(\zeta, \eta)$ for $\mathrm{GSp}(2n, q)$ with different n and q. See the table for $\mathrm{GSp}(6, 3)$ as an instance. Note that the sum of the entries in this table is 18341406720, which is exactly the order of $\mathrm{GSp}(6, 3)$, as expected.

GSp(6, 3)	$C(\zeta,\eta)$	$\eta = 0$	$\eta = 1$	$\eta = 2$
	$\zeta = 1$	3053423790	3058639785	3058639785
	$\zeta = 2$	3063934512	3053384424	3053384424

GSp(4, 5)	$C(\zeta,\eta)$	$\eta = 0$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
	$\zeta = 1$	3867500	3713125	3713125	3713125	3713125
	$\zeta = 2$	0	0	0	0	0
	$\zeta = 3$	0	0	0	0	0
	$\zeta = 4$	3870000	3712500	3712500	3712500	3712500

GSp(6, 5)	$C(\zeta,\eta)$	$\eta = 0$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
	$\zeta = 1$	91408007812500	91395326171875	91401669921875	91401669921875	91395326171875
	$\zeta = 2$	91395312500000	91408015625000	91395328125000	91395328125000	91408015625000
	$\zeta = 3$	91395312500000	91395328125000	91408015625000	91408015625000	91395328125000
	$\zeta = 4$	91408007812500	91401669921875	91395326171875	91395326171875	91401669921875

References

[1] D. S. Kim, *Gauss sums for symplectic groups over a finite field*, Monatsh. Math. **126** (1998).

[2] D. S. Kim, *Exponential sums for symplectic groups and their applications*, Acta Arith. **88** (1999).